H

millennia institute

2016 Preliminary Examination II
Pre-University 3

MATHEMATICS **9740/02**

Paper 2 **21 September 2016**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This question paper consists of 7 printed pages.

Section A: Pure Mathematics [40 marks]

1 **(i)** Given that $\ln y = \sin^{-1}(2x)$, show that $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\sqrt{\left(1-4x^2\right)} = 2y$. [2]

By further differentiation of this result, find the Maclaurin series for y up to and including the term in x^2. [3]

(ii) Hence find an approximate value of $\int_0^{\frac{1}{3}} 5\mathrm{e}^{\sin^{-1}(2x)}\,\mathrm{d}x$. [2]

2 The function f is defined by

$$\mathrm{f} : x \mapsto x^2 + \lambda x - \lambda^2, \quad x \in \mathbb{R}, \ x \leq -\frac{\lambda}{2},$$

where λ is a positive real number.

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain and range of f^{-1}. [4]

The function g is defined by

$$\mathrm{g} : x \mapsto \mathrm{e}^{x}, \quad x \in \mathbb{R}.$$

(ii) Explain why the composite function fg does not exist. [2]

(iii) Find the composite function gf in a similar form and find its range. [3]

3 **(i)** Show that the substitution $w = x^2 y$ reduces the differential equation

$$x^3\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + x^2\left(1-2y\right)$$

to

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{1}{x} + x.$$

Hence find the general solution of y in terms of x. [5]

(ii) Find the particular solution of the differential equation for which $y = 1$ when $x = 1$. [1]

(iii) What can you say about the value of y for every solution curve as $x \to \infty$? [1]

(iv) Sketch, on a single diagram, for $x > 0$, the solution curve found in part **(ii)**, together with two other members of the family of solution curves. [You do not need to find the x-intercepts and turning points.] [3]

4 The line l_1 has equation $x-1=\frac{y}{-2}=\frac{z+1}{3}$, and the plane p_1 has equation $2x+y=2$.

(i) Find a vector equation of l_1 and hence show that it lies in p_1. [3]

(ii) A line l_2 has equation $\mathbf{r}=\begin{pmatrix}5\\0\\0\end{pmatrix}+\mu\begin{pmatrix}1\\0\\t\end{pmatrix}$ where $\mu\in\mathbb{R}$ and t is a real constant. Given that the angle between l_2 and p_1 is 30°, find the possible values of t. [2]

It is given that the plane p_2 has equation $\mathbf{r}\bullet\begin{pmatrix}1\\s\\-3\end{pmatrix}=4$ and p_1 and p_2 meet in a line l_1.

(iii) By considering the normal vectors of p_1 and p_2, show that $s=-4$. [2]

(iv) Point A has coordinates $(1,\ 1,\ 2)$. Find the foot of perpendicular from A to p_2. Hence find the shortest distance between A and p_2. [5]

The plane p_3 has equation $x+y+z-\alpha(2x-y+1)=\beta$.

(v) Given that the three planes p_1, p_2 and p_3 have no point in common, what can be said about the values of α and β? [2]

Section B: Statistics [60 marks]

5 An organisation has 1000 employees. The ratio of the number of its employees undertaking the administrative, professional and managerial positions is 2 : 5 : 1. A list of the employees in alphabetical order by name is generated. A sample of 50 employees is to be chosen to take part in an emergency exercise to assess their level of preparedness in handling an emergency.

(i) Describe how a stratified sample of 50 employees could be chosen. [2]

(ii) Explain a disadvantage of systematic sampling as compared to stratified sampling, in the context of the question. [1]

6 The continuous random variable X has the distribution $\mathrm{N}(\mu,\sigma^2)$. It is known that $\mathrm{P}(X<85)=\mathrm{P}(X>101)=0.0548$. Calculate the values of μ and σ. [3]

7 Customers at a supermarket pay for their purchases either by cash or credit card. On average, the probability that a randomly chosen customer pays by credit card is p . In a random sample of 10 customers, the number of customers who pay by credit card is denoted by X .

(i) State, in the context of this question, an assumption needed to model X by a binomial distribution. [1]

Assume now that X has the distribution $\mathrm{B}(10, p)$.

(ii) Given that the probability that all customers pay by credit card is 0.268, show that the value of p is 0.877. [2]

(iii) Find the most likely number of customers who pay by credit card. [1]

Use a suitable approximation to find the probability that, in a random sample of 100 customers, there are at least 80 customers who pay by credit card. [3]

8 A player in a bowling team is assessed through three rolls of the ball. The probability that he bowls a strike in the first roll is 0.6. From the second roll onwards, the probability that he bowls a strike is:

- p if he bowls a strike in the preceding game,
- 0.7 if he did not bowl a strike in the preceding game.

(i) Construct a tree diagram to represent all the possible outcomes. [2]

The probability of him bowling exactly one strike at the end of his 3 games is 0.1415.

(ii) Find p. [3]

(iii) Find the probability of him bowling a strike for his last game if he wins at most one strike at the end of 3 rolls. [3]

9 **(a)** Six pairs of values of variables x and y are measured. Draw a sketch of a possible scatter diagram of the data for the case in which the product moment correlation coefficient is approximately zero. [1]

(b) Research is being carried out to study how the population of a particular species of bacteria in millions (y) varies with temperature in degree Celsius (x). Observations at different temperatures give the data shown in the following table.

x	−4.1	−3.2	−1.8	−0.7	0.1	1.1	2.0	3.1	3.8
y	11.5	15.0	18.2	20.4	20.1	19.4	17.8	15.0	12.0

(i) Draw a scatter diagram for these values, labelling the axes. [1]

It is thought that the population at different temperatures can be modelled by one of the formulae

$$y = ab^x \quad \text{or} \quad y = cx^2 + d\,,$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the values of the product moment correlation coefficient between

(A) x and $\ln y$

(B) x^2 and y [2]

(iii) Explain which of the formulae above is the better model. [1]

(iv) By finding the equation of a suitable regression line for the model in part **(iii)**, estimate the temperature(s), correct to 1 decimal place, that corresponds to the bacteria population of 19 million. Comment on the reliability of this value. [3]

10 **(a)** A four-digit number (larger than 999) is randomly written on a piece of paper. What is the probability that the first and last digits are the same? [1]

(b) Five male guests each bring their female partners to their annual dinner and dance. They are first required to stand in two rows for photo-taking in which four people stand in front and the remaining six people stand at the back.

(i) Find the number of ways to arrange all of them without restrictions. [1]

(ii) Find the probability in which *not* all the male guests are standing next to each other. [3]

(iii) They are then asked to sit at a round table in which there are ten numbered seats. Find the probability that each male guest and his female partner sit together. [2]

Three of the ten guests are randomly selected to take a group photograph. Find the probability that exactly two of them are males. [2]

11 A company supplies butter in small packets which have masses that are normally distributed. The mass of butter in a packet is denoted by X grams. The company claims that the mean mass of butter in a packet is 10 grams. The masses of a random sample of 12 packets of butter are summarised by

$$\sum(x-20)=-118.8, \quad \sum(x-20)^2=1176.4.$$

(i) Calculate the unbiased estimates of the population mean and variance. [2]

(ii) Test at the 10% level of significance whether there is any evidence to doubt the company's claim. [4]

The population variance of X is now known to be 0.15 grams^2. Another random sample of n packets is taken, and it has a mean of 9.9 grams. A test at the $2\frac{1}{2}\%$ significance level is carried out on this sample, and it is concluded that the company's claim is overstated. Obtain an inequality involving n, and hence find the set of values that n can take. [4]

12 An insurance company deals with two types of insurance, travel insurance and non-travel insurance. The number of travel insurance claims received by an insurance company per week follows a Poisson distribution with mean 5.6.

(i) Find the probability that, in a randomly chosen week, at least 5 travel insurance claims are received. [1]

(ii) Given 50 random one-week periods, estimate the probability that the mean number of travel insurance claims received per one-week period is at most 5. [3]

(iii) Find the probability that, in a random sample of 10 weeks, fewer than 3 weeks have at least 5 travel insurance claims received per week. [2]

The number of non-travel insurance claims received by the same insurance company per week follows a Poisson distribution with mean 7.7. The number of claims received for the two types of insurance claims are independent.

(iv) Find the probability that in a randomly chosen week, the total number of insurance claims received is 10. [2]

(v) Taking a month to be 4 weeks, use suitable approximations to find the probability that the total number of travel insurance claims over 2 independent months is more than twice the number of non-travel insurance claims in a month. [3]

(vi) Explain why the Poisson distribution model may not be a good model for the number of travel insurance claims received in a year. [1]

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