H2

TAMPINES JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

MATHEMATICS 9740/01

Paper 1

Tuesday, 13 Sep 2016

3 hours

Additional Materials: Answer paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in, including the Cover Page.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 A curve C has equation $\mathrm{e}^{x+y} + \mathrm{e} = (3y+1)^2$.

(i) By considering $\frac{\mathrm{d}y}{\mathrm{d}x}$, show that C has no stationary points. [5]

(ii) Write down an equation relating x and y at which the tangent is parallel to the y-axis. [1]

2 Referred to the origin O, the points A and B have position vectors given by $\mathbf{a} = \begin{pmatrix} \cos t \\ -\sin t \\ 0.5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \cos 2t \\ \sin 2t \\ -1 \end{pmatrix}$ respectively, where t is a real parameter such that $0 \le t < \pi$.

(i) Show that $\mathbf{a} \cdot \mathbf{b} = p + \cos(qt)$, where p and q are constants to be determined. [2]

(ii) Hence find the exact value of t for which $\angle AOB$ is a maximum. [3]

3 **(i)** Describe a sequence of transformations that will transform the curve with equation $y = \frac{1}{x^2}$ on to the curve with equation $y = \frac{4}{(x-1)^2}$. [2]

(ii) It is given that

$$\mathrm{f}(x) = \begin{cases} x+2 & \text{for } 0 < x \le 2, \\ \dfrac{4}{(x-1)^2} & \text{for } 2 < x \le 3, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+3)$ for all real values of x.

Sketch the graph of $y = \mathrm{f}(x)$ for $-2 \le x \le 6$. [3]

4 **Do not use a calculator in answering this question.**

One root of the equation $z^3 + az^2 + bz + 15 = 0$, where a and b are real, is $z = 1 + 2\mathrm{i}$.

(i) Write down the other complex root. [1]

(ii) Explain why the cubic equation must have one real root. [1]

(iii) Find the value of the real root and the values of a and b. [5]

5 Functions f and g are defined by

$$\mathrm{f} : x \mapsto \frac{3x-1}{3x-3}, \quad x \in \mathbb{R},\ x < 1,$$
$$\mathrm{g} : x \mapsto \sqrt{x-2}, \quad x \in \mathbb{R},\ 2 \le x < 3.$$

(i) Find $\mathrm{f}^{-1}(x)$. [2]

(ii) Show that $\mathrm{f}^2(x) = x$. Hence find the exact value of $\mathrm{f}^{2017}(0)$. [2]

(iii) Show that the composite function fg exists. Find an expression for $\mathrm{fg}(x)$ and state the domain and range of fg. [5]

6 It is given that $\mathrm{f}(x) = \dfrac{x+3}{(1-x)^n}$, where $-1 < x < 1$ and n is a positive integer.

(i) Find the binomial expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^2. [4]

(ii) Given that the coefficient of x^2 in the above expansion is 21, find the value of n. [3]

(iii) Given now that $n = 2$, by substituting a suitable value of x into the expansion in part **(i)**, find the exact value of $\displaystyle\sum_{r=1}^{\infty} \frac{4r-1}{4^{r-1}}$. [3]

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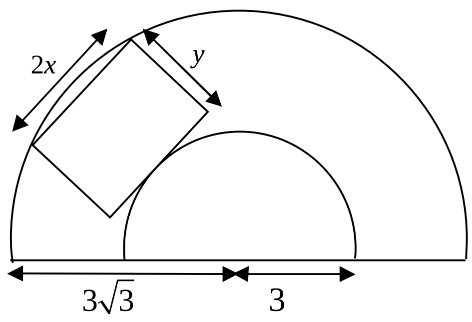

The figure shows a rectangular sheet of length $2x$ metres and breadth y metres to be placed in a horizontal position along a garden walkway bounded by low vertical fence of which a horizontal cross-section is two concentric semicircles of radii 3 metres and $3\sqrt{3}$ metres. One side of the sheet of length $2x$ metres must be tangential to the inner fence, and the two ends of the opposite side must touch the outer fence, as shown in the figure. The rectangular sheet is assumed to have negligible thickness.

(i) By finding x^2 in terms of y, show that the area A square metres of the sheet, is given by $A = 2\sqrt{18y^2 - 6y^3 - y^4}$. [3]

(ii) Use differentiation to find, the maximum value of A, proving that it is a maximum. [7]

8 The plane p_1 passes through the points $A(4,1,1)$ and $B(2,1,0)$ and is parallel to the vector $4\mathbf{i}-\mathbf{j}+2\mathbf{k}$. A line l has equation $\dfrac{x-2}{-2}=y+1=z+4$.

(i) Show that a vector perpendicular to the plane p_1 is parallel to $\mathbf{i}-2\mathbf{k}$. Find the equation of p_1 in scalar product form. [3]

(ii) Find the coordinates of the point C at which l intersects p_1. [3]

(iii) The point D with coordinates $(2, -1, -4)$ lies on l. Find the position vector of the foot of the perpendicular from D to p_1. Find the coordinates of the point E which is the mirror image of D in p_1. [4]

The plane p_2 contains the line l and the point A.

(iv) The planes p_1 and p_2 meet in a line m. Find a vector equation for m. [2]

9 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 1 \quad \text{and} \quad 3u_{n+1} = 2u_n - 1 \text{ for } n \geq 1 .$$

(i) Use the method of mathematical induction to prove that

$$u_n = 3\left(\frac{2}{3}\right)^n - 1.$$ [5]

(ii) Find $\sum_{r=1}^{n} u_r$. [3]

(iii) Give a reason why the series $\sum_{r=1}^{n}(u_r+1)$ converges, and write down the value of the sum to infinity. [2]

(iv) Explain, with the aid of a sketch, whether the value of $\sum_{r=1}^{\infty}(u_r+1)$ is an overestimation or underestimation of the value of $\int_0^{\infty} 3\left(\frac{2}{3}\right)^x \mathrm{d}x$. [2]

10 The mass, x grams, of a certain substance present in a chemical reaction at time t minutes satisfies the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(4 + 2x - x^2),$$

where $0 \le x \le 1$ and k is a constant. It is given that $x = 1$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{2}$ when $t = 0$.

(i) Show that $k = -\frac{1}{10}$. [1]

(ii) By first expressing $4 + 2x - x^2$ in completed square form, find t in terms of x. [5]

(iii) Hence find the time taken for there to be none of the substance present in the chemical reaction, giving your answer correct to 3 decimal places. [1]

(iv) Express the solution of the differential equation in the form $x = \mathrm{f}(t)$ and sketch the part of the curve with this equation which is relevant in this context. [5]

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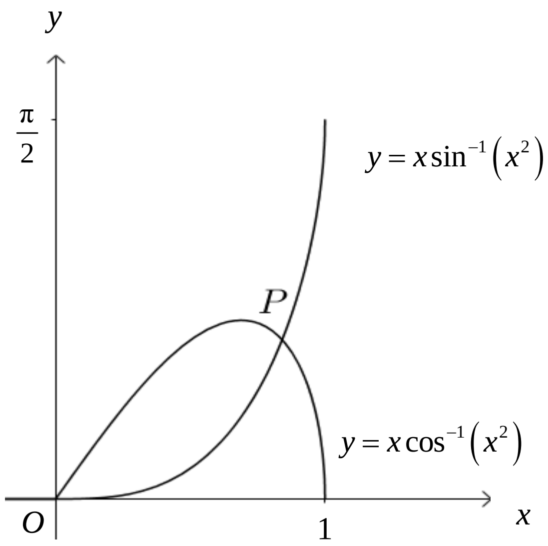

The diagram shows the curves with equations $y = x\sin^{-1}(x^2)$ and $y = x\cos^{-1}(x^2)$, where $0 \le x \le 1$. The curves meet at the point P with coordinates $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{4}\left(\frac{1}{\sqrt[4]{2}}\right)\right)$.

(i) Find the derivative of $\sqrt{1-x^4}$. [1]

(ii) Find the exact value of the area of the region bounded by the two curves. [5]

(iii) The region bounded by the curve $y = x\sin^{-1}(x^2)$, the line $y = \frac{\pi}{4}\left(\frac{1}{\sqrt[4]{2}}\right)$ and the y-axis is rotated about the y-axis through $360°$. By considering the parametric equations

$$x = t \quad \text{and} \quad y = t\sin^{-1}(t^2),$$

show that the volume of the solid formed is given by

$$\pi\int_0^{\frac{1}{\sqrt[4]{2}}} \left[\frac{2t^4}{\sqrt{1-t^4}} + t^2\sin^{-1}(t^2)\right] dt.$$ [3]

(iv) Hence find the volume of the solid formed when the region bounded by the curve $y = x\sin^{-1}(x^2)$ and the line $y = \frac{\pi}{4}x$ is rotated through $360°$ about the y-axis. Give your answer correct to 5 significant figures. [3]

End of Paper